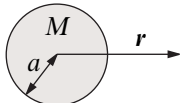


3.3 Gravitation

Newtonian gravitation

Newton's law of gravitation	$\mathbf{F}_1 = \frac{Gm_1m_2}{r_{12}^2} \hat{\mathbf{r}}_{12} \quad (3.40)$	$m_{1,2}$ masses $\mathbf{F}_1$ force on m_1 ($= -\mathbf{F}_2$) $\mathbf{r}_{12}$ vector from m_1 to m_2 $\hat{}$ unit vector
Newtonian field equations ^a	$\mathbf{g} = -\nabla\phi \quad (3.41)$ $\nabla^2\phi = -\nabla \cdot \mathbf{g} = 4\pi G\rho \quad (3.42)$	G constant of gravitation $\mathbf{g}$ gravitational field strength ϕ gravitational potential ρ mass density
Fields from an isolated uniform sphere, mass M , $\mathbf{r}$ from the centre	$\mathbf{g}(\mathbf{r}) = \begin{cases} -\frac{GM}{r^2} \hat{\mathbf{r}} & (r > a) \\ -\frac{GM}{a^3} \mathbf{r} & (r < a) \end{cases} \quad (3.43)$	$\mathbf{r}$ vector from sphere centre M mass of sphere a radius of sphere
	$\phi(\mathbf{r}) = \begin{cases} -\frac{GM}{r} & (r > a) \\ \frac{GM}{2a^3} (r^2 - 3a^2) & (r < a) \end{cases} \quad (3.44)$	

^aThe gravitational force on a mass m is $m\mathbf{g}$.

General relativity^a

Line element	$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = -d\tau^2$	(3.45)	ds dτ $g_{\mu\nu}$ dx^μ $\Gamma^\alpha_{\beta\gamma}$ $_{,\alpha}$ $;\alpha$ ϕ A^α B_α	invariant interval proper time interval metric tensor differential of x^μ Christoffel symbols partial diff. w.r.t. x^α covariant diff. w.r.t. x^α scalar contravariant vector covariant vector
Christoffel symbols and covariant differentiation	$\Gamma^\alpha_{\beta\gamma} = \frac{1}{2} g^{\alpha\delta} (g_{\delta\beta,\gamma} + g_{\delta\gamma,\beta} - g_{\beta\gamma,\delta})$ $\phi_{;\gamma} = \phi_{,\gamma} \equiv \partial\phi/\partial x^\gamma$ $A^\alpha_{;\gamma} = A^\alpha_{,\gamma} + \Gamma^\alpha_{\beta\gamma} A^\beta$ $B_{\alpha;\gamma} = B_{\alpha,\gamma} - \Gamma^\beta_{\alpha\gamma} B_\beta$	(3.46) (3.47) (3.48) (3.49)		
Riemann tensor	$R^\alpha_{\beta\gamma\delta} = \Gamma^\alpha_{\mu\gamma} \Gamma^\mu_{\beta\delta} - \Gamma^\alpha_{\mu\delta} \Gamma^\mu_{\beta\gamma} + \Gamma^\alpha_{\beta\delta,\gamma} - \Gamma^\alpha_{\beta\gamma,\delta}$ $B_{\mu;\alpha;\beta} - B_{\mu;\beta;\alpha} = R^\gamma_{\mu\alpha\beta} B_\gamma$ $R_{\alpha\beta\gamma\delta} = -R_{\alpha\delta\beta\gamma} ; \quad R_{\beta\alpha\gamma\delta} = -R_{\alpha\beta\gamma\delta}$ $R_{\alpha\beta\gamma\delta} + R_{\alpha\delta\beta\gamma} + R_{\alpha\gamma\delta\beta} = 0$	(3.50) (3.51) (3.52) (3.53)	$R^\alpha_{\beta\gamma\delta}$	Riemann tensor
Geodesic equation	$\frac{Dv^\mu}{D\lambda} = 0$ where $\frac{DA^\mu}{D\lambda} \equiv \frac{dA^\mu}{d\lambda} + \Gamma^\mu_{\alpha\beta} A^\alpha v^\beta$	(3.54) (3.55)	v^μ λ	tangent vector (= $dx^\mu/d\lambda$) affine parameter (e.g., τ for material particles)
Geodesic deviation	$\frac{D^2 \xi^\mu}{D\lambda^2} = -R^\mu_{\alpha\beta\gamma} v^\alpha \xi^\beta v^\gamma$	(3.56)	ξ^μ	geodesic deviation
Ricci tensor	$R_{\alpha\beta} \equiv R^\sigma_{\alpha\sigma\beta} = g^{\sigma\delta} R_{\delta\alpha\sigma\beta} = R_{\beta\alpha}$	(3.57)	$R_{\alpha\beta}$	Ricci tensor
Einstein tensor	$G^{\mu\nu} = R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R$	(3.58)	$G^{\mu\nu}$ R	Einstein tensor Ricci scalar (= $g^{\mu\nu} R_{\mu\nu}$)
Einstein's field equations	$G^{\mu\nu} = 8\pi T^{\mu\nu}$	(3.59)	$T^{\mu\nu}$ p	stress-energy tensor pressure (in rest frame)
Perfect fluid	$T^{\mu\nu} = (p + \rho) u^\mu u^\nu + p g^{\mu\nu}$	(3.60)	ρ u^ν	density (in rest frame) fluid four-velocity
Schwarzschild solution (exterior)	$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)$	(3.61)	M (r, θ, ϕ) t	spherically symmetric mass (see page 183) spherical polar coords. time
Kerr solution (outside a spinning black hole)	$ds^2 = -\frac{\Delta - a^2 \sin^2\theta}{\rho^2} dt^2 - 2a \frac{2Mr \sin^2\theta}{\rho^2} dt d\phi + \frac{(r^2 + a^2)^2 - a^2 \Delta \sin^2\theta}{\rho^2} \sin^2\theta d\phi^2 + \frac{\rho^2}{\Delta} dr^2 + \rho^2 d\theta^2$	(3.62)	J a Δ ρ^2	angular momentum (along z) $\equiv J/M$ $\equiv r^2 - 2Mr + a^2$ $\equiv r^2 + a^2 \cos^2\theta$

^aGeneral relativity conventionally uses the $(-1, 1, 1, 1)$ metric signature and “geometrized units” in which $G = 1$ and $c = 1$. Thus, $1 \text{ kg} = 7.425 \times 10^{-28} \text{ m}$ etc. Contravariant indices are written as superscripts and covariant indices as subscripts. Note also that ds^2 means $(ds)^2$ etc.

